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Coherent states approach to Penning trap

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Abstract
By using a matrix technique, which allows us to identify directly the ladder operators, the Penning trap coherent states are derived as eigenstates of the appropriate annihilation operators. These states are compared with those obtained through the displacement operator. The associated wavefunctions and mean values for some relevant operators in these states are also evaluated. It turns out that the Penning trap coherent states minimize the Heisenberg uncertainty relation.

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1. Introduction

The coherent states (CS) approach to quantum physical systems [1–3] constitutes nowadays an alternative to the standard method, which addresses the same problem in terms of energy eigenstates and eigenvalues. For years the CS have been derived for plenty of Hamiltonians having either a ground or a top state, and some of them admit a group theoretical construction in which this state is acted on by an appropriate displacement operator [2, 4]. However, there exist interesting physical systems for which the Hamiltonians have neither ground nor top state [5, 6], but a systematic technique to build up the corresponding CS is required anyway. One of those systems consists of a charged particle in an ideal Penning trap [7, 8]. Such an arrangement, sometimes called a geonium atom, has been largely used to perform high-precision measurements of fundamental properties of particles [7]. Moreover, it could be used to test and/or control some intrinsically quantum phenomena as entanglement, decoherence, wavepacket reduction, etc [8–10].

In this paper we are going to address, from a CS viewpoint, the quantum motion of a charged particle in a Penning trap. With this aim, in section 2 we will present some generalities of the standard coherent states. In sections 3 and 4, we will introduce the Penning trap Hamiltonian and discuss its corresponding algebraic structure. It will be shown that the system possesses a certain 'extremal' state, which plays the role of a ground state although there is no minimum energy eigenvalue. In section 5 we will construct the wavefunction associated with the extremal state, while in section 6 we will perform the corresponding CS construction. The mean values of some physical quantities in the CS will be calculated in section 7. Finally, in section 8 our conclusions will be presented.

2. Standard coherent states

Glauber definitions of CS are based on the properties of the harmonic oscillator [11], which have been applied to several different systems (see, e.g., [1–3]):

(1) The CS $|z\rangle$ are eigenstates of the annihilation operator a:

$$a|z\rangle = z|z\rangle, \qquad z \in \mathbb{C}. \tag{1}$$

(2) They arise from acting the displacement operator on the ground state:

$$|z\rangle = D(z)|\psi_0\rangle, \qquad D(z) = \exp(za^\dagger - z^* a), \tag{2}$$

with $a^\dagger$ being the creation operator.

(3) The CS satisfy the minimum Heisenberg uncertainty relation for X and P:

$$(\Delta X)_z(\Delta P)_z = \hbar/2, \tag{3}$$

where $(\Delta\mathcal{O})_z^2 = \langle z|(\mathcal{O} - \langle\mathcal{O}\rangle_z)^2|z\rangle = \langle\mathcal{O}^2\rangle_z - \langle\mathcal{O}\rangle_z^2$ is the mean-square deviation of an observable $\mathcal{O}$ in the state $|z\rangle$.

It is worth noting an additional property of the standard CS, which is relevant since some authors consider it as the fourth CS definition. It is the completeness relationship $\frac{1}{\pi}\int |z\rangle\langle z|\, \mathrm{d}^2 z = \mathbf{1}$, where $\mathbf{1}$ is the identity operator. In fact, the CS are overcomplete in the sense that for any convergent sequence of complex numbers z_n the corresponding CS $|z_n\rangle$ form a complete set [12].

For systems different from the harmonic oscillator, these definitions lead to different sets of CS. In this paper, we will use the first and second definitions to find the CS for a charged particle in an ideal Penning trap; we will show that they satisfy as well equation (3).

3. Penning trap Hamiltonian and the matrix $\boldsymbol{\Lambda}$

Let us consider a spinless particle of mass m and electric charge e inside an ideal Penning trap, i.e., under the influence of a constant homogeneous magnetic field pointing along the z-direction $\vec{B} = B\hat{k}$, and a static electric field $\vec{E} = -\vec{\nabla}\Phi(\vec{r})$, both arising from the following vector and quadrupole scalar potentials:

$$\vec{A}(\vec{r}) = -\frac{1}{2}\vec{r} \times \vec{B}, \qquad \Phi(\vec{r}) = \frac{\Phi_0}{d^2}(x^2 + y^2 - 2z^2). \tag{4}$$

Throughout this paper, the small letters $\vec{r}, \vec{p}, x, y, z, p_x, p_y, p_x$ will denote either classical coordinates and momenta or the eigenvalues associated with the corresponding quantum operators, the last ones being represented by capital letters $\vec{R}, \vec{P}, X, Y, Z, P_x, P_y, P_z$. The Hamiltonian describing our quantum system is given by

$$H = \frac{1}{2m}\left(\vec{P} - \frac{e}{c}\vec{A}(\vec{R})\right)^2 + e\Phi(\vec{R}) = \frac{\vec{P}^2}{2m} + bL_z + \frac{m}{2}[(b^2 + v)(X^2 + Y^2) - 2vZ^2], \tag{5}$$

where $\vec{L} = \vec{R} \times \vec{P}$ is the angular momentum operator, $b = -\frac{eB}{2mc}$, $v = \frac{2e\Phi_0}{md^2}$ and we take by simplicity $b > 0$. To ensure that the particle is trapped inside the cavity, some restrictions on the parameters b, v have to be taken: first of all $v < 0$ in order that the z-motion is bounded (so that this mode is characterized by a standard oscillator Hamiltonian). However, the corresponding repulsive oscillators in the x–y plane do not have to destroy the trapped motion induced by the magnetic field, which is achieved by taking $b^2 + v > 0$.

From now on we will assume that $m = 1$ and $\hbar = 1$. Note that this assumption is equivalent to the following procedure: (i) first making the operator changes $\hat{R}_i = R_i\sqrt{m/\hbar}$, $\hat{P}_i = P_i/\sqrt{m\hbar}$, $i = 1, 2, 3$, $\hat{H} = H/\hbar$; (ii) then dropping the hats in order to simplify the notation. Thus, the Hamiltonian we are dealing with reads

$$H = \frac{\vec{P}^2}{2} + bL_z + \frac{1}{2}[(b^2+v)(X^2+Y^2) - 2vZ^2], \tag{6}$$

where $[R_i, P_j] = \mathrm{i}\delta_{ij}$.

It is useful to work in the Heisenberg picture in which the evolution of the operator vector $\eta(t) = U^\dagger(t)\eta U(t)$ is simply determined from a matrix equation:

$$\frac{\mathrm{d}\eta(t)}{\mathrm{d}t} = U^\dagger(t)[\mathrm{i}H, \eta]U(t) = U^\dagger(t)\mathbf{\Lambda}\eta U(t) = \mathbf{\Lambda}\eta(t) \quad \Rightarrow \quad \eta(t) = \mathrm{e}^{\mathbf{\Lambda}t}\eta, \tag{7}$$

where $\eta = (\vec{R}, \vec{P})^{\mathrm{T}}$ involves the observables $\vec{R}$, $\vec{P}$ in the Schrödinger picture, the superindex $^{\mathrm{T}}$ denotes to transpose the involved vector, $U(t)$ is the evolution operator such that $U(0) = \mathbf{1}$. The calculation of $[\mathrm{i}H, \eta] = \mathbf{\Lambda}\eta$ leads to

$$\mathbf{\Lambda} = \begin{pmatrix} 0 & -b & 0 & 1 & 0 & 0 \\ b & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -b^2-v & 0 & 0 & 0 & -b & 0 \\ 0 & -b^2-v & 0 & b & 0 & 0 \\ 0 & 0 & 2v & 0 & 0 & 0 \end{pmatrix}. \tag{8}$$

Let us find next the right (u) and left (f) eigenvectors of the matrix $\mathbf{\Lambda}$, which are called eigenvectors and eigenforms, respectively. Since $\mathbf{\Lambda}$ is non-Hermitian, the eigenforms f are not necessarily the adjoints of the eigenvectors u. In order to determine both, we solve in the first place the characteristic equation of $\mathbf{\Lambda}$:

$$|\mathbf{\Lambda} - \lambda\mathbf{1}| = \lambda^6 + 4b^2\lambda^4 - v(8b^2+3v)\lambda^2 - 2v^3 = 0. \tag{9}$$

Thus, the eigenvalues are $\pm\lambda_1 = \pm\mathrm{i}\omega_1$, $\pm\lambda_2 = \pm\mathrm{i}\omega_2$, $\pm\lambda_3 = \pm\mathrm{i}\omega_3$, where

$$\omega_1 = b + \sqrt{b^2+v}, \qquad \omega_2 = b - \sqrt{b^2+v}, \qquad \omega_3 = \sqrt{-2v}. \tag{10}$$

We label as u_k, u_k^* and f_k, f_k^* the eigenvectors and eigenforms associated with the eigenvalues $\lambda_k, \lambda_k^* = -\lambda_k$, respectively, i.e., $\mathbf{\Lambda}u_k = \lambda_k u_k$, $\mathbf{\Lambda}u_k^* = -\lambda_k u_k^*$, $f_k\mathbf{\Lambda} = \lambda_k f_k$, $f_k^*\mathbf{\Lambda} = -\lambda_k f_k^*$, $k = 1, 2, 3$, with the * denoting complex conjugation. An explicit calculation leads to

$$u_1 = s_1\left(\frac{1}{\sqrt{b^2+v}}, \frac{-\mathrm{i}}{\sqrt{b^2+v}}, 0, \mathrm{i}, 1, 0\right)^{\mathrm{T}}, \qquad f_1 = t_1\left(\sqrt{b^2+v}, \mathrm{i}\sqrt{b^2+v}, 0, -\mathrm{i}, 1, 0\right),$$

$$u_2 = s_2\left(\frac{-1}{\sqrt{b^2+v}}, \frac{\mathrm{i}}{\sqrt{b^2+v}}, 0, \mathrm{i}, 1, 0\right)^{\mathrm{T}}, \qquad f_2 = t_2\left(-\sqrt{b^2+v}, -\mathrm{i}\sqrt{b^2+v}, 0, -\mathrm{i}, 1, 0\right),$$

$$u_3 = s_3\left(0, 0, \frac{-\mathrm{i}}{\sqrt{-2v}}, 0, 0, 1\right)^{\mathrm{T}}, \qquad f_3 = t_3\left(0, 0, \mathrm{i}\sqrt{-2v}, 0, 0, 1\right),$$

where $s_j, t_j \in \mathbb{C}$, $j = 1, 2, 3$. We require that the eigenvectors and eigenforms be dual to each other [5, 6, 13], namely, $f_j u_k = f_j^* u_k^* = \delta_{jk}$, $f_j u_k^* = f_j^* u_k = 0$, implying that $s_1 = \frac{1}{4t_1}$, $s_2 = \frac{1}{4t_2}$, $s_3 = \frac{1}{2t_3}$. The constants t_j will be fixed later to simplify some commutation relationships. Finally, the eigenvectors and eigenforms satisfy the unit matrix decomposition

$$\mathbf{1} = \sum_{k=1}^{3}(u_k \otimes f_k + u_k^* \otimes f_k^*) \quad \Rightarrow \quad \mathbf{\Lambda} = \sum_{k=1}^{3}\lambda_k(u_k \otimes f_k - u_k^* \otimes f_k^*), \tag{11}$$

with $\otimes$ denoting the tensor product. The $\boldsymbol{\Lambda}$-expression in (11) allows us to decompose the Heisenberg trajectories as three oscillating modes of frequencies ω_j [5, 6]. Moreover, it will characterize as well the algebraic structure of the Hamiltonian.

4. Algebraic structure of *H*

We can define now three pairs of ladder operators of H, $L_k = f_k^*\eta$, $L_k^\dagger = f_k\eta$, $k = 1, 2, 3$, which obey the following commutation relations with H:

$$[H, L_k] = -\mathrm{i}f_k^*[\mathrm{i}H, \eta] = -\omega_k L_k, \qquad \left[H, L_k^\dagger\right] = \omega_k L_k^\dagger. \tag{12}$$

An explicit calculation leads to

$$\begin{aligned} &L_1 = t_1^*\left[\sqrt{b^2+v}(X - \mathrm{i}Y) + \mathrm{i}(P_x - \mathrm{i}P_y)\right], \\ &L_2 = t_2^*[-\sqrt{b^2+v}(X - \mathrm{i}Y) + \mathrm{i}(P_x - \mathrm{i}P_y)], \qquad L_3 = t_3^*(-\mathrm{i}\sqrt{-2v}Z + P_z). \end{aligned} \tag{13}$$

By evaluating next the commutators between $L_i, L_j^\dagger$, the following non-null results are obtained:

$$\begin{aligned} &\left[L_1, L_1^\dagger\right] = 2|t_1|^2(\omega_1 - \omega_2) = 1, \\ &\left[L_2, L_2^\dagger\right] = -2|t_2|^2(\omega_1 - \omega_2) = -1, \qquad \left[L_3, L_3^\dagger\right] = 2|t_3|^2\omega_3 = 1, \end{aligned} \tag{14}$$

where we have finally chosen $t_i \in \mathbb{R}^+$ such that $t_1 = t_2 = 1/\sqrt{2(\omega_1 - \omega_2)}$, $t_3 = 1/\sqrt{2\omega_3}$ to simplify at maximum equation (14). On the other hand, $\left[L_i^\dagger, L_j^\dagger\right] = [L_i, L_j] = 0$, $i, j = 1, 2, 3$.

Now H is factorized in terms of $L_k, L_k^\dagger$ as follows [5, 6]:

$$H = \omega_1 L_1^\dagger L_1 - \omega_2 L_2 L_2^\dagger + \omega_3 L_3^\dagger L_3 + (\omega_1 - \omega_2 + \omega_3)/2. \tag{15}$$

Moreover, equations (14) and (15) allow us to identify three independent oscillator modes for H, each one characterized by its number N_k, annihilation B_k and creation $B_k^\dagger$ operator, in the way

$$N_k = B_k^\dagger B_k, \quad k = 1, 2, 3, \tag{16}$$

$$B_1 = L_1, \quad B_2 = L_2^\dagger, \quad B_3 = L_3, \quad B_1^\dagger = L_1^\dagger, \quad B_2^\dagger = L_2, \quad B_3^\dagger = L_3^\dagger. \tag{17}$$

They obey the standard commutation relations:

$$[N_k, B_k] = -B_k, \quad \left[N_k, B_k^\dagger\right] = B_k^\dagger, \quad \left[B_j, B_k^\dagger\right] = \delta_{jk}, \qquad j, k = 1, 2, 3. \tag{18}$$

Hence, one can construct a basis $\{|n_1, n_2, n_3\rangle, n_j = 0, 1, 2, \ldots, j = 1, 2, 3\}$ of common eigenstates of N_1, N_2, N_3,

$$N_j|n_1, n_2, n_3\rangle = n_j|n_1, n_2, n_3\rangle, \qquad j = 1, 2, 3, \tag{19}$$

departing from an *extremal state* $|0, 0, 0\rangle$ which is annihilated by B_1, B_2, B_3:

$$B_j|0, 0, 0\rangle = 0, \qquad j = 1, 2, 3. \tag{20}$$

If we assume that $|0, 0, 0\rangle$ is normalized, it turns out that [14]

$$|n_1, n_2, n_3\rangle = (n_1!n_2!n_3!)^{-1/2} B_1^{\dagger n_1} B_2^{\dagger n_2} B_3^{\dagger n_3}|0, 0, 0\rangle. \tag{21}$$

Moreover, $B_j, B_j^\dagger$, $j = 1, 2, 3$, act onto $|n_1, n_2, n_3\rangle$ in a standard way:

$$B_1|n_1, n_2, n_3\rangle = \sqrt{n_1}|n_1 - 1, n_2, n_3\rangle, \qquad B_1^\dagger|n_1, n_2, n_3\rangle = \sqrt{n_1 + 1}|n_1 + 1, n_2, n_3\rangle,$$

and similar expressions for the action of B_2, $B_2^\dagger$, B_3, $B_3^\dagger$. Note that $|n_1, n_2, n_3\rangle$ is the eigenstate of the Penning trap Hamiltonian with the eigenvalue $E_{n_1,n_2,n_3} = \omega_1(n_1 + 1/2) - \omega_2(n_2 + 1/2) + \omega_3(n_3 + 1/2) \equiv E(n_1, n_2, n_3)$. In particular, the extremal state $|0, 0, 0\rangle$ has eigenvalue $E_{0,0,0} = (\omega_1 - \omega_2 + \omega_3)/2$, i.e., it is neither a ground nor a top state since its energy is 'in the middle' of the spectrum of H. Following [15], it is seen that there is an *intrinsic* algebraic structure for our system, which is characterized by a linear relationship between the Penning trap Hamiltonian H and the three number operators N_k:

$$H = E(N_1, N_2, N_3) = \omega_1 N_1 - \omega_2 N_2 + \omega_3 N_3 + E_{0,0,0}. \tag{22}$$

As it happens for one-dimensional systems, in our three-dimensional example the detailed structure is contained in the operator relation (22), which is responsible for the specific spectrum and, consequently, for the lack of a ground or a top proper energy. On the other hand, the global structure comes from the very existence of the three independent oscillator modes for H, each one characterized by the standard generators $\{N_j, B_j, B_j^\dagger\}$, $j = 1, 2, 3$. This global behavior allows us to identify in a natural way the extremal state $|0, 0, 0\rangle$ which, although is neither a ground nor a top energy eigenstate, plays the same role as the ground state for the one-dimensional harmonic oscillator.

5. Extremal state wavefunction

The existence of the extremal state $|0, 0, 0\rangle$ is guaranteed by a theorem which is proven elsewhere [5]. It ensures that, if the operators

$$B_j = \mathrm{i}\vec{P}\cdot\vec{\alpha}_j + \vec{R}\cdot\vec{\beta}_j, \qquad B_j^\dagger = -\mathrm{i}\vec{\alpha}_j^\dagger\cdot\vec{P} + \vec{\beta}_j^\dagger\cdot\vec{R}, \qquad j = 1, 2, 3, \tag{23}$$

obey the commutation relations (18), then the system of partial differential equations $\langle\vec{r}|B_j|0, 0, 0\rangle = 0$, $j = 1, 2, 3$, for the extremal state wavefunction $\phi_{\mathbf{0}}(\vec{r}) \equiv \langle\vec{r}|0, 0, 0\rangle$ has a square integrable solution given by

$$\phi_{\mathbf{0}}(\vec{r}) = c\exp\left(-\tfrac{1}{2}a_{ij}r_i r_j\right) = c\exp\left(-\tfrac{1}{2}\vec{r}^{\mathrm{T}}\mathbf{a}\vec{r}\right), \tag{24}$$

where $\mathbf{a} = (a_{ij})$ is a complex symmetric matrix satisfying

$$\mathbf{a}\vec{\alpha}_j = \vec{\beta}_j, \qquad j = 1, 2, 3. \tag{25}$$

According to (23), through equations (13) and (17) we identify the vectors

$$\begin{aligned}
\vec{\alpha}_1 &= \frac{1}{2(b^2+v)^{1/4}}(1, -\mathrm{i}, 0)^{\mathrm{T}}, & \vec{\beta}_1 &= (b^2+v)^{1/2}\vec{\alpha}_1, \\
\vec{\alpha}_2 &= -\frac{1}{2(b^2+v)^{1/4}}(1, \mathrm{i}, 0)^{\mathrm{T}}, & \vec{\beta}_2 &= (b^2+v)^{1/2}\vec{\alpha}_2, \\
\vec{\alpha}_3 &= -\frac{\mathrm{i}}{\sqrt{2}(-2v)^{1/4}}(0, 0, 1)^{\mathrm{T}}, & \vec{\beta}_3 &= (-2v)^{1/2}\vec{\alpha}_3.
\end{aligned} \tag{26}$$

Thus, $\mathbf{a} = \mathrm{diag}[\sqrt{b^2+v}, \sqrt{b^2+v}, \sqrt{-2v}]$, and from (24) we finally get the extremal state wavefunction we were looking for:

$$\phi_{\mathbf{0}}(\vec{r}) = c\exp\left(-\frac{\sqrt{b^2+v}}{2}(x^2+y^2) - \sqrt{\frac{-v}{2}}z^2\right). \tag{27}$$

6. Penning trap coherent states

Once the Penning trap Hamiltonian has been expressed appropriately in terms of annihilation and creation operators, we can develop a similar treatment as for the harmonic oscillator to build up the corresponding coherent states.

6.1. Annihilation operator coherent states

In the first place let us look for the annihilation operator coherent states (AOCS) as the common eigenstates of B_1, B_2, B_3:

$$B_j|z_1, z_2, z_3\rangle = z_j|z_1, z_2, z_3\rangle, \qquad j = 1, 2, 3. \tag{28}$$

Following a standard procedure, let us expand them in the basis $\{|n_1, n_2, n_3\rangle\}$:

$$|z_1, z_2, z_3\rangle = \sum_{n_1,n_2,n_3=0}^{\infty} c_{n_1,n_2,n_3}|n_1, n_2, n_3\rangle. \tag{29}$$

By asking that (28) is satisfied, three recurrence relationships for c_{n_1,n_2,n_3} will be obtained, which in turn lead to the following expressions:

$$c_{n_1,n_2,n_3} = (n_1!)^{-1/2} z_1^{n_1} c_{0,n_2,n_3} = (n_2!)^{-1/2} z_2^{n_2} c_{n_1,0,n_3} = (n_3!)^{-1/2} z_3^{n_3} c_{n_1,n_2,0}. \tag{30}$$

Hence, it is straightforward to show that

$$c_{n_1,n_2,n_3} = (n_1!n_2!n_3!)^{-1/2} z_1^{n_1} z_2^{n_2} z_3^{n_3} c_{0,0,0}, \tag{31}$$

where $c_{0,0,0}$ is to be found from the normalization condition. Thus, up to a global phase factor, the normalized AOCS become finally

$$|z_1, z_2, z_3\rangle = \mathrm{e}^{-\frac{|z_1|^2+|z_2|^2+|z_3|^2}{2}} \sum_{n_1,n_2,n_3=0}^{\infty} (n_1!n_2!n_3!)^{-1/2} z_1^{n_1} z_2^{n_2} z_3^{n_3}|n_1, n_2, n_3\rangle. \tag{32}$$

6.2. Displacement operator coherent states

According to equation (2), for the jth mode of the Penning trap Hamiltonian we have to take into account the corresponding displacement operator $D_j(z_j) = \exp\left(z_j B_j^\dagger - z_j^* B_j\right)$. By using the BCH formula it turns out that

$$D_j(z_j) = \mathrm{e}^{-\frac{|z_j|^2}{2}} \mathrm{e}^{z_j B_j^\dagger} \mathrm{e}^{-z_j^* B_j}, \qquad j = 1, 2, 3. \tag{33}$$

Now, the global displacement operator is given by

$$D(\mathbf{z}) \equiv D(z_1, z_2, z_3) = D_1(z_1)D_2(z_2)D_3(z_3), \tag{34}$$

where $\mathbf{z}$ denotes the complex variables z_1, z_2, z_3 associated with the three modes. By employing now the second definition, we get the displacement operator coherent states (DOCS) $|\mathbf{z}\rangle$ from applying $D(\mathbf{z})$ to the extremal state $|0, 0, 0\rangle$:

$$|\mathbf{z}\rangle = D(\mathbf{z})|0, 0, 0\rangle = \mathrm{e}^{-\frac{|z_1|^2+|z_2|^2+|z_3|^2}{2}} \sum_{n_1,n_2,n_3=0}^{\infty} \frac{z_1^{n_1} z_2^{n_2} z_3^{n_3}|n_1, n_2, n_3\rangle}{\sqrt{n_1!n_2!n_3!}}. \tag{35}$$

By comparing (32) and (35) we realize that the DOCS and the AOCS are the same. Moreover, since $\left[z_j B_j^\dagger - z_j^* B_j, z_k B_k^\dagger - z_k^* B_k\right] = 0$, $j, k = 1, 2, 3$, we get

$$\begin{aligned} D(\mathbf{z}) &= \exp\left(z_1 B_1^\dagger + z_2 B_2^\dagger + z_3 B_3^\dagger - z_1^* B_1 - z_2^* B_2 - z_3^* B_3\right) = \exp[\mathrm{i}(\vec\Sigma\cdot\vec R - \vec\Gamma\cdot\vec P)] \\ &= C(\mathbf{z})F(\vec R)\exp(-\mathrm{i}\vec\Gamma\cdot\vec P) = [C(\mathbf{z})]^{-1}\exp(-\mathrm{i}\vec\Gamma\cdot\vec P)F(\vec R), \end{aligned} \tag{36}$$

where we have used the BCH formula and equation (23) to identify

$$\begin{aligned} \vec\Gamma &= \begin{pmatrix} (b^2+v)^{-\frac14}\,\mathrm{Re}[z_1 - z_2] \\ -(b^2+v)^{-\frac14}\,\mathrm{Im}[z_1 + z_2] \\ -(-v/2)^{-\frac14}\,\mathrm{Im}[z_3] \end{pmatrix}, \qquad \vec\Sigma = \begin{pmatrix} (b^2+v)^{\frac14}\,\mathrm{Im}[z_1 - z_2] \\ (b^2+v)^{\frac14}\,\mathrm{Re}[z_1 + z_2] \\ (-8v)^{\frac14}\,\mathrm{Re}[z_3] \end{pmatrix}, \\ C(\mathbf{z}) &= \mathrm{e}^{-\mathrm{i}\vec\Gamma\cdot\vec\Sigma/2} = \exp\{\mathrm{i}(\mathrm{Re}[z_1]\,\mathrm{Im}[z_2] + \mathrm{Re}[z_2]\,\mathrm{Im}[z_1] + \mathrm{Re}[z_3]\,\mathrm{Im}[z_3])\}, \\ F(\vec R) &= \mathrm{e}^{\mathrm{i}\vec\Sigma\cdot\vec R} = \exp\{\mathrm{i}(b^2+v)^{\frac14}(\mathrm{Im}[z_1-z_2]X + \mathrm{Re}[z_1+z_2]Y) + \mathrm{i}(-8v)^{\frac14}\,\mathrm{Re}[z_3]Z\}. \end{aligned} \tag{37}$$

Since the operator $e^{-i\vec{P}\cdot\vec{\Gamma}}$, $\Gamma_i \in \mathbb{R}$, performs a coordinate displacement in the way $\langle\vec{r}|\, e^{-i\vec{P}\cdot\vec{\Gamma}} = \langle\vec{r}-\vec{\Gamma}|$, we finally get

$$\phi_{\mathbf{z}}(\vec{r}) \equiv \langle\vec{r}|\mathbf{z}\rangle = \langle\vec{r}|D(\mathbf{z})|0,0,0\rangle = C(\mathbf{z})F(\vec{r})\langle\vec{r}|\,e^{-i\vec{P}\cdot\vec{\Gamma}}|0,0,0\rangle,$$
$$= C(\mathbf{z})F(\vec{r})\phi_{\mathbf{0}}\left(x - \frac{\mathrm{Re}[z_1-z_2]}{(b^2+v)^{\frac14}},\, y + \frac{\mathrm{Im}[z_1+z_2]}{(b^2+v)^{\frac14}},\, z + \left(\frac{-2}{v}\right)^{\frac14}\mathrm{Im}[z_3]\right), \qquad (38)$$

with $\phi_{\mathbf{0}}(\vec{r})$ given by (27).

7. Mean values of physical quantities

Let us evaluate next the mean values $\langle R_j\rangle_{\mathbf{z}} \equiv \langle\mathbf{z}|R_j|\mathbf{z}\rangle$, $\langle P_j\rangle_{\mathbf{z}} \equiv \langle\mathbf{z}|P_j|\mathbf{z}\rangle$, $j = 1,2,3$, and the corresponding mean-square deviations in a given CS $|\mathbf{z}\rangle$. To do that, we analyze first how the operators R_j, R_j^2, P_j, P_j^2 are transformed under $D(\mathbf{z})$. By using equations (36) and (37) it is straightforward to show that

$$D^\dagger(\mathbf{z})R_j^n D(\mathbf{z}) = (R_j+\Gamma_j)^n, \qquad D^\dagger(\mathbf{z})P_j^n D(\mathbf{z}) = (P_j+\Sigma_j)^n, \qquad n = 1,2,\ldots \qquad (39)$$

Therefore,

$$\langle R_j\rangle_{\mathbf{z}} = \langle R_j\rangle_{\mathbf{0}} + \Gamma_j, \qquad \left\langle R_j^2\right\rangle_{\mathbf{z}} = \left\langle R_j^2\right\rangle_{\mathbf{0}} + 2\Gamma_j\langle R_j\rangle_{\mathbf{0}} + \Gamma_j{}^2, \qquad (\Delta R_j)_{\mathbf{z}}^2 = (\Delta R_j)_{\mathbf{0}}^2, \qquad (40)$$
$$\langle P_j\rangle_{\mathbf{z}} = \langle P_j\rangle_{\mathbf{0}} + \Sigma_j, \qquad \left\langle P_j^2\right\rangle_{\mathbf{z}} = \left\langle P_j^2\right\rangle_{\mathbf{0}} + 2\Sigma_j\langle P_j\rangle_{\mathbf{0}} + \Sigma_j{}^2, \qquad (\Delta P_j)_{\mathbf{z}}^2 = (\Delta P_j)_{\mathbf{0}}^2. \qquad (41)$$

Note that the mean-square deviations of R_j and P_j are independent of z_1, z_2, z_3 but depend on $\langle R_j\rangle_{\mathbf{0}}$, $\langle P_j\rangle_{\mathbf{0}}$, $\left\langle R_j^2\right\rangle_{\mathbf{0}}$, $\left\langle P_j^2\right\rangle_{\mathbf{0}}$, $j = 1,2,3$, which need to be evaluated. The first six quantities can be obtained from the homogeneous equations $\langle B_k\rangle_{\mathbf{0}} = i(\vec{\alpha}_k)_j\langle P_j\rangle_{\mathbf{0}} + (\vec{\beta}_k)_j\langle R_j\rangle_{\mathbf{0}} = 0$, $\left\langle B_k^\dagger\right\rangle_{\mathbf{0}} = -i(\vec{\alpha}_k^*)_j\langle P_j\rangle_{\mathbf{0}} + (\vec{\beta}_k^*)_j\langle R_j\rangle_{\mathbf{0}} = 0$, $k = 1,2,3$ (see (23) and use that $B_k|0,0,0\rangle = \langle 0,0,0|B_k^\dagger = 0$). By using (26), the system to be solved becomes

$$-i\sqrt{-2v}\langle Z\rangle_{\mathbf{0}} + \langle P_z\rangle_{\mathbf{0}} = 0,$$
$$\sqrt{b^2+v}(\langle X\rangle_{\mathbf{0}} - i\langle Y\rangle_{\mathbf{0}}) + i(\langle P_x\rangle_{\mathbf{0}} - i\langle P_y\rangle_{\mathbf{0}}) = 0,$$
$$-\sqrt{b^2+v}(\langle X\rangle_{\mathbf{0}} + i\langle Y\rangle_{\mathbf{0}}) - i(\langle P_x\rangle_{\mathbf{0}} + i\langle P_y\rangle_{\mathbf{0}}) = 0,$$

and the complex conjugate equations. Its solution is given by

$$\langle R_j\rangle_{\mathbf{0}} = \langle P_j\rangle_{\mathbf{0}} = 0, \qquad j = 1,2,3. \qquad (42)$$

In order to obtain $\left\langle R_j^2\right\rangle_{\mathbf{0}}$, $\left\langle P_j^2\right\rangle_{\mathbf{0}}$, we calculate the mean values for the several products of pairs involving B_j, $B_k^\dagger$. From these 36 equations just 21 are linearly independent: $\langle B_jB_k\rangle_{\mathbf{0}} = 0$, $j = 1,2,3$, $k \leqslant j$ (six equations); $\left\langle B_j^\dagger B_k^\dagger\right\rangle_{\mathbf{0}} = 0$, $j = 1,2,3$, $k \leqslant j$ (six equations); $\left\langle B_k^\dagger B_j\right\rangle_{\mathbf{0}} = 0$, $j,k = 1,2,3$, (nine equations). By solving this linear system, the non-null mean values of the 21 independent products of R_i and P_j are now

$$\langle X^2\rangle_{\mathbf{0}} = \langle Y^2\rangle_{\mathbf{0}} = [4(b^2+v)]^{-\frac12}, \qquad \langle Z^2\rangle_{\mathbf{0}} = (-8v)^{-\frac12},$$
$$\left\langle P_x^2\right\rangle_{\mathbf{0}} = \left\langle P_y^2\right\rangle_{\mathbf{0}} = [(b^2+v)/4]^{\frac12}, \qquad \left\langle P_z^2\right\rangle_{\mathbf{0}} = (-v/2)^{\frac12},$$
$$\langle XP_x\rangle_{\mathbf{0}} = \langle YP_y\rangle_{\mathbf{0}} = \langle ZP_z\rangle_{\mathbf{0}} = i/2.$$

The previous formulae imply that equations (40) and (41) become

$$(\Delta X)_{\mathbf{z}}^2 = (\Delta Y)_{\mathbf{z}}^2 = [4(b^2+v)]^{-\frac12}, \qquad (\Delta Z)_{\mathbf{z}}^2 = (-8v)^{-\frac12},$$
$$(\Delta P_x)_{\mathbf{z}}^2 = (\Delta P_y)_{\mathbf{z}}^2 = [(b^2+v)/4]^{\frac12}, \qquad (\Delta P_z)_{\mathbf{z}}^2 = (-v/2)^{\frac12},$$

and therefore

$$(\Delta X)_{\mathbf{z}}(\Delta P_x)_{\mathbf{z}} = (\Delta Y)_{\mathbf{z}}(\Delta P_y)_{\mathbf{z}} = (\Delta Z)_{\mathbf{z}}(\Delta P_z)_{\mathbf{z}} = 1/2.$$

This means that our CS have minimum Heisenberg uncertainty relations.

Finally, by using equations (16) and (22) we calculate the mean value of the Hamiltonian H in a given CS $|\mathbf{z}\rangle$:

$$\langle H\rangle_{\mathbf{z}} = \omega_1|z_1|^2 - \omega_2|z_2|^2 + \omega_3|z_3|^2 + E_{0,0,0}. \tag{43}$$

A similar calculation for $\langle H^2\rangle_{\mathbf{z}}$ can be done, leading to

$$(\Delta H)^2_{\mathbf{z}} = \left(b + \sqrt{b^2 + v}\right)^2|z_1|^2 + \left(b - \sqrt{b^2 + v}\right)^2|z_2|^2 - 2v|z_3|^2. \tag{44}$$

Once again, the fact that H is not positive definite is clearly reflected in (43).

Along this work we have assumed that $b = -\frac{eB}{2mc} > 0$. For $b < 0$, small differences concerning the identification of the appropriate annihilation and creation operators arise. However, the extremal state and CS wavefunctions $\phi_{\mathbf{0}}(\vec{r})$, $\phi_{\mathbf{z}}(\vec{r})$ as well as the corresponding mean values will coincide with those previously calculated. In particular, the Heisenberg uncertainty relation will achieve once again its minimum value [14].

8. Concluding remarks

In this paper a technique to find the CS for a charged particle in an ideal Penning trap was introduced. We have shown that the coherent states, calculated through both definitions given by equations (1) and (2), are the same. We introduced also a prescription to obtain the mean values of several physical observables in a given coherent state. We have found, finally, that the Penning trap coherent states (derived algebraically) obey also the third CS definition, i.e., they satisfy the minimum Heisenberg uncertainty relation.

Let us remark that the method presented here is quite general, and it could be applied to other systems characterized by quadratic Hamiltonians. In order to implement systematically this treatment, we have to identify first the stability regions where the non-degenerate eigenvalues of $\mathbf{\Lambda}$ become purely imaginary, which ensures that the Heisenberg and classical trajectories are trapped. In the trap regime, the Hamiltonian is decomposed in terms of independent harmonic oscillators, and thus our procedure can be straightforwardly applied. Note that generalizations of this kind have been elaborated elsewhere (see, e.g., [16]). However, in our method it is direct to identify the global sign accompanying each individual oscillator involved in the Hamiltonian decomposition. As we saw in our Penning trap example, those signs determine the existence or not of a ground state for the system, a fact which is not well known in the literature. Moreover, they become fundamental for the determination of the intrinsic algebraic structure of the involved Hamiltonian (compare equation (22)). Observe that some of these properties were found previously for operators imitating the Hamiltonian in non-inertial reference frames [5, 6]. By means of this example we have shown that such a property arises as well for Hamiltonians in inertial frames of reference.

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